Sibbald, T. M. (2015). Math exploration using the Morley triangle. *Ontario Association for Mathematics Education Gazette, 54*(2), 31–34.

▲ Math Exploration Using the Morley Triangle

TIM SIBBALD
EMAIL: timothys@nipissingu.ca

Tim is an Assistant Professor of mathematics education in the Schulich School of Education, Nipissing University. He has previously taught high school and been a mathematician in industry. He currently serves as the President of the OAME.

The Morley triangle is composed within a triangle by trisecting each angle and identifying the three points where neighbouring tri-sectors intersect (see Figure 1). The Morley triangle, ΔDEF, is an equilateral triangle. This discovery was made around 1900 and, unlike many high school geometry theorems, was unknown in antiquity. This article explores ways to bring this relatively modern geometry result to life in the classroom. The focus is primarily numerical because the triangle can be constructed and investigated using the trisection option in GeoGebra (software freely available at **www.geogebra.org**) and the spreadsheet facility that records values can be used for experimental investigation. The challenge that this paper addresses is how to relate the position, size, and orientation of the Morley triangle to the original triangle. The issue of proof is not addressed because it would be difficult for students to achieve in a constructive manner without significant guidance (several proofs can be found online at **www.cut-the-knot.com/triangle/Morley/**).

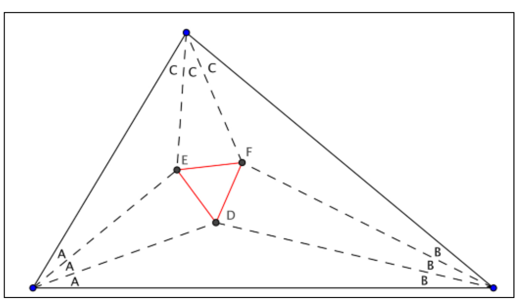

Figure 1. *The Morley triangle.*

Before reading further, consider that the orientation of this article is toward having students create questions about the Morley triangle that they can investigate numerically. Teachers will need to gauge their comfort level with doing this, but should consider asking open questions to promote student exploration. Consider asking your students to each make a triangle and have them measure the angles with protractors, then construct the trisectors and the Morley triangle. When I have done this, a sense of disbelief and awe have arisen. The idea can be adapted to creating a class set of triangles that follow a pattern (such as a common base and angles of 10, 20, 30,… degrees in one corner), where construction of Morley triangles allows for a gallery walk to see how the pattern among the original triangles becomes a pattern among the Morley triangles. Hypotheses may arise and discussion is likely with support from the teacher.

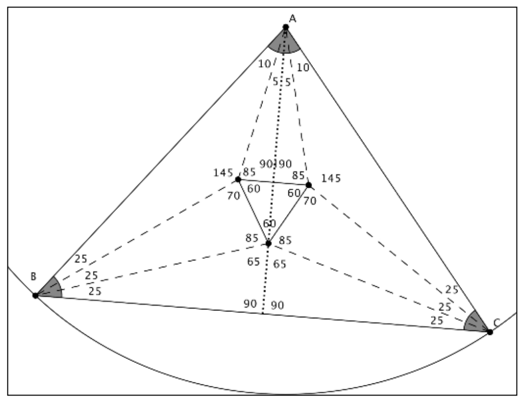

Figure 2. *Angle determination exercise.*

One case that can be addressed geometrically in Grades 9 and 10 is the Morley triangle constructed within an isosceles triangle. In Grade 9, students can work out the size of all the angles, given the size of one of the isosceles triangle angles. Consider the example in Figure 2 of an isosceles triangle with a unique angle of 30°, where students are to use the fact that the central triangle is the Morley equilateral triangle. In Grade 10 and above, trigonometry allows for considerably more detail. Consider the layout in Figure 3, where the dotted line AD is the line of symmetry. The angle x is the independent variable, and the challenge for trigonometry students is to determine h and m that define the location of the Morley triangle. Note that in the isosceles triangle case, the orientation of Morley's triangle is not considered because it shares the axis of symmetry, and GF is parallel to BC.

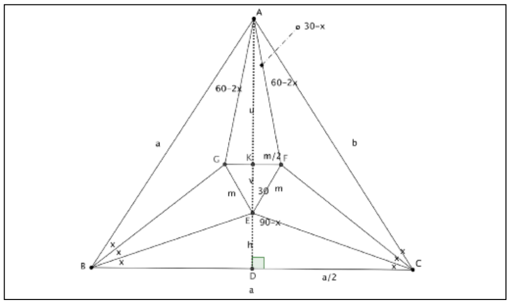

Figure 3. *The general isosceles Morley triangle.*

In the following examination, one could define $a = 1$ as one way to simplify the algebra for students. In Figure 3, triangle CDE gives the relationship defining h as:

$$h = \frac{a\tan(x)}{2}$$

Since the central triangle, EGF, is equilateral, $v = \frac{\sqrt{3}m}{2}$, and in triangle AKF, the relationship relating u to m is:

$$\tan(30 - x) = \frac{m}{2u}$$

$$u = \frac{m}{2\tan(30 - x)}$$

The relevance of this is that $u + v + h$ is the height of the outer triangle. It also allows substitution based on the three previous equations and leads to:

$$u + v + h = b^2 - \frac{a^2}{4} = \frac{m}{2\tan(30 - x)} + \frac{\sqrt{3}m}{2} + \frac{a\tan(x)}{2}$$

Using some algebraic equation solving, which will require teacher support, the relationship defining m in terms of a and x is:

$$m = \frac{\left(4b^2 - a^2 - 2a\tan(x)\right)\tan(30 - x)}{2 + 2\sqrt{3}\tan(30 - x)}$$

That is the extent of theoretical considerations for students in the high school environment. A little more will be stated at the end of the article for teachers who are interested in a more detailed look at the algebra behind this phenomenon. For the classroom, the focus becomes numerical in order to keep it accessible to students.

A Numerical Approach

When I first played with the Morley triangle, I was using Geometer's Sketchpad®, which had no facility for trisecting angles. This required explicitly calculating points corresponding to the trisections. It was feasible and I did manage it, but I never felt it was reasonable to suggest for other math teachers. That changed when I found that GeoGebra includes a tool for making angles with a specified size ("angle of a given size") and that the specification can be in terms of any other measured angle. In other words, trisecting angles is an option.

To make numerical investigation achievable, GeoGebra includes a spreadsheet. Any variable can be recorded to the spreadsheet (right click, "record to spreadsheet"). In the version of GeoGebra I am using (version 5.0.119.0), the recording of angles is in degrees only with the endless list option. Angles are in radians if a finite list is selected. Two additional details need to be recognized. Firstly, calculations in the spreadsheet should be done when recording of variables is turned off (red dots in column labels need to be clicked with the mouse). Second, if you wish to work in degrees, then calculations need to be done in GeoGebra, rather than copying the spreadsheet data to other software. If data is copied and pasted into another spreadsheet, it registers as text with the degree sign included; numerical calculation is no longer feasible. If you want your students to work with the values in other software, it requires recording a finite list and guiding your students to make a calculated column that converts the radians into degrees (i.e. , multiply by 180/π). Luckily, angles measured in degrees are fine for inspection and for manually using the values for graphing by hand or duplicating in a graphing calculator.

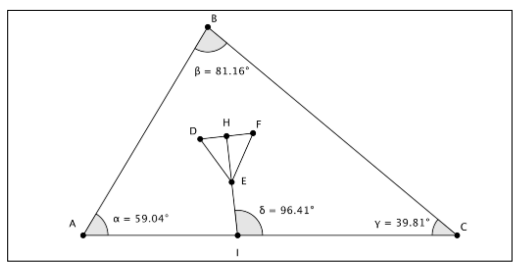

Figure 4. *Experimental set-up for measuring orientation.*

As a first numerical experiment, consider asking students to measure the orientation of the Morley triangle. Students can make a Morley triangle, or you can provide a template (see "concluding remarks" for accessing a template) to suit the amount of time you have for the activity. An experimental set-up for measuring the orientation is provided in Figure 4. Students should be guided to question how to measure the orientation, and encouraged to explore other approaches—Figure 4 is only one approach! Students might, for example, extend DE down to AC and measure the angle that is formed. Alternatively, they could measure DF relative to the horizontal by adding a line through D that is parallel to AC. The option shown uses a DE to construct a perpendicular bisector (HE) of the Morley triangle and extends it to the base of the original triangle (point I). The angle HIC then measures the orientation. Right clicking on each angle in GeoGebra allows recording to the spreadsheet. Forewarn students not to drag too fast; it is computationally intensive and GeoGebra does periodically run into difficulties—make sure students save their experimental set-up before any dragging that records to the spreadsheet. When I ran the experiment, I acquired the results shown in the screen shot in Figure 5. Note that the jumps in values in the spreadsheet correspond to computation being unable to keep up with the speed I dragged point B.

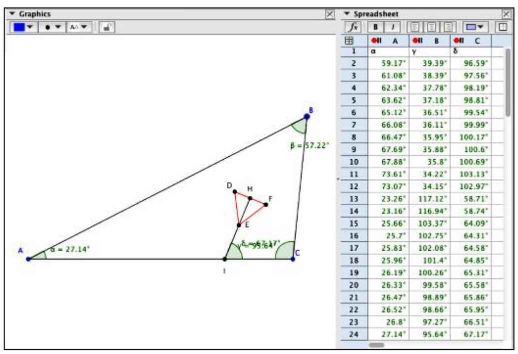

Figure 5. *Screenshot of orientation experiment.*

Following the data collection, students can turn the recording feature off by clicking the red "record" buttons at the top of each column in the spreadsheet. They can then create hypotheses by visually inspecting the columns and observing trends. This can lead them on a path of organized experimentation, where they constrain the point that is dragged. They may, for example, have the dragged point, B, affixed to a line that is parallel to AC. What would happen then? This helps tailor hypotheses and can be subsequently used for discussion in the classroom.

A second type of experiment is beneficial and suited to anyone wishing to have data that can easily be copied and pasted to other software. Have students design their own experiment involving only lengths. They could examine how the Morley perimeter varies with the triangle perimeter. Measuring areas would also work. To demonstrate what is feasible, this experiment will follow a hunch and see how the side length of the Morley triangle is related to the radius of the circumscribed circle for the triangle. The experimental set-up is shown in Figure 6. In the figure, the circumcentre is found as the intersection of the lines perpendicular to the midpoint of each side (point K). The circumcentre is the middle of a circle that passes through the three vertices (A, B, and C) of the triangle. The radius of the circumcircle is KC, which has been measured, while the side length of the Morley triangle is *m*.

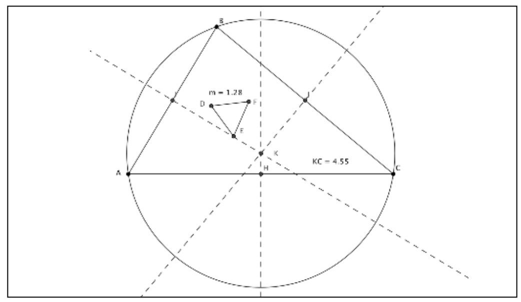

Figure 6. *Experiment to investigate lengths.*

Recording the data took multiple tries, and it might have been wise to reduce the amount of software running on my computer. However, when it was successful, I clicked on the red circles to turn off recording to the spreadsheet and that then calculated the ratio column on the spreadsheet. The ratio column is the ratio of the circumcentre radius to the Morley side length. I confess, I initially thought the value always seemed to be between three and four, and thought that I should see if I could defy that finding. The outcome of the experiment is shown in Figure 7.

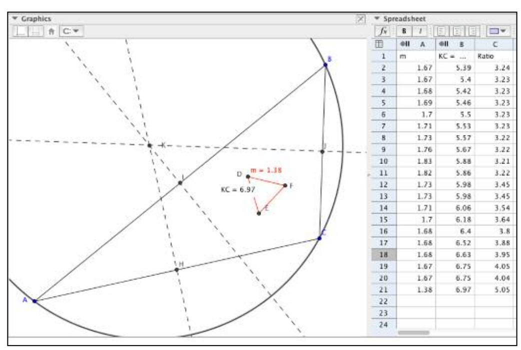

Figure 7. *Results of experiment concerning lengths.*

It was through some serious play that it became evident that as the largest angle in the triangle grew large, the circumcircle also grew large, but the Morley triangle shrank. Encourage your students to go the extra step in explaining: The Morley triangle is contained in the original triangle and is equilateral, so in some sense, it is confined by the minimum linear size of the triangle. In other words, the height of the obtuse triangle is becoming much smaller, and that reflects a constraint on the size of the Morley triangle. Perhaps that is not quite as clear as you would like? That is because there is a whole new experiment lurking that asks: How big is the largest equilateral triangle that fits inside a given triangle?

Why not elaborate on the algebra?

As mentioned earlier, this paper purposefully pursued a numerical approach. If you are wondering why, then consider a particular arrangement shown in Figure 8, where the triangle has been placed on a coordinate grid so that the three points are on the axes of the coordinate system. The three points are (-a, 0), (b, 0) , and (0, c). While this naming approach does not match the "standard" approach of having small letter sides opposite capital letter angles, it does aid clarity by having the coordinate a affiliated with the angle A. The coordinate arrangement, with the three points on the axes, is convenient for clarifying why theory can only be used to some extent; the triangle will be assumed to be acute in this case.

Within this set-up, the coordinates of point E, the Morley point closest to the x-axis, can be determined. In Figure 8, the tangent of angle A and tangent of angle B provide two equations in two unknowns. The two equations are:

$$\tan(A) = \frac{y}{a+x} \quad \tan(B) = \frac{y}{b-x}$$

When solved, the two coordinates are:

$$x = \frac{b\tan(B) - a\tan(A)}{\tan(B) + \tan(A)}$$

$$y = (a+x)\tan(A) = \frac{(a+b)\tan(A)\tan(B)}{\tan(A)+\tan(B)}$$

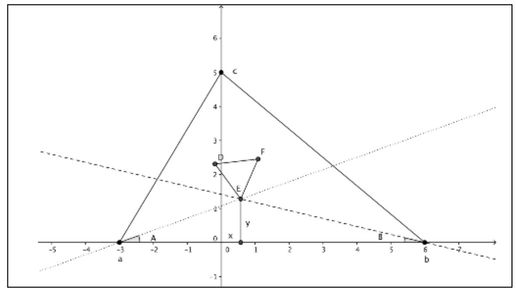

Figure 8. *An analytic geometry set-up.*

This resolves point *E* in terms of *a*, *b*, *A*, and *B*. To remove references to the angles requires resolution of tan(A) in terms of side lengths of the right triangle in the second quadrant. However, since *A* is one-third of the full angle, determining tan(A) requires solving the single-angle tangent in terms of the triple-angle tangent, which can then be replaced with a ratio $\left(\frac{c}{a}\right)$. Given that:

$$\tan(3A) = \frac{c}{a} = \frac{3\tan(A) - \tan^3(A)}{1 - 3\tan^2(A)}$$

What is important is to realize that this involves solving a cubic equation for tan(A). In general, that is beyond the theoretical scope of the high school curriculum, except using numerical means.

Similarly, tan(2A) can be expressed as a ratio of values based on coordinates in the Morley triangle. This provides quadratic algebraic connections between the coordinates, but does not provide any clear way to separate the variables in order to use high school approaches to quadratics.

There is quite a rich vein of mathematics that teachers could explore further, with an eye to including it in advanced functions. However, it is not as readily suited to teaching as the numerical approach. The remarks here are intended to make it clear that using methods that are not numerical will pose significant pedagogical challenges. I remain hopeful that others will develop these aspects.

Concluding Remarks

Thank you to the two reviewers who provided some good advice for this article. In particular, a template file and a file corresponding to Figure 6 are available at (**http://faculty.nipissingu.ca/timothys**) and also at the GeoGebra Institute of Canada website (**www.geogebracanada.org/home/resources**). ▲